Explicit Folded Reed–Solomon and Multiplicity Codes Achieve Near-Optimal List Decodability

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Talk at Peking University (Online)
March 18th, 2024

Joint Work With Yeyuan Chen at University of Michigan

Talk Overview

- Motivations of Error-Correcting Codes

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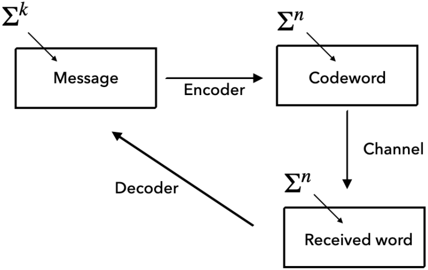
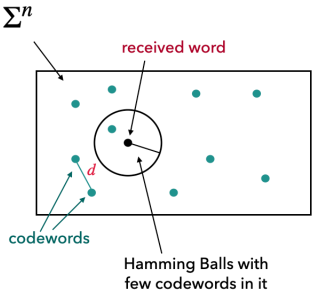
- Future Directions

Motivations of Error-Correcting Codes





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$$d(C) := \min_{x \neq y \in C} d(x, y),$$

where $d : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{N}$ is the Hamming distance.

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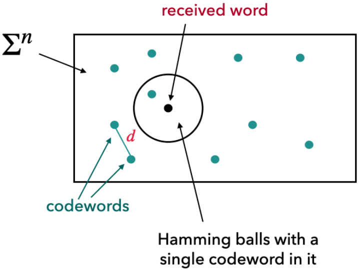
- $d(x, z) \leq d(x, y) + d(y, z) \longrightarrow$ for any $y \in \Sigma^n$, we have

$$\left| B_{\left\lfloor \frac{d(C)-1}{2} \right\rfloor}(y) \cap C \right| \leq 1.$$

Unique Decoding: Backgrounds

Unique decoding: transmitted codeword $x \in C$ and received word $y \in \Sigma^n$, if $d(x,y) \le \left\lfloor \frac{\mathrm{d}(C)-1}{2} \right\rfloor$, then $x = B_{\left\lfloor \frac{d(C)-1}{2} \right\rfloor}(y) \cap C$



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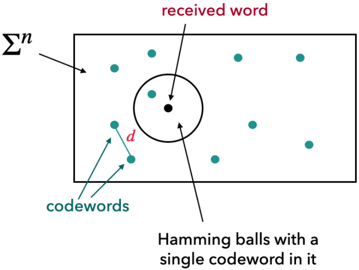
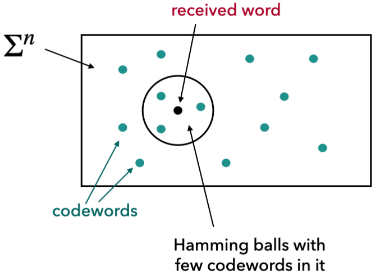


Question: More than $\left\lfloor \frac{\mathrm{d}(C)-1}{2} \right\rfloor$?

List decoding (introduced by Elias 1957, Wozencraft 1958): Given transmitted codeword $c \in C$ and received word $y \in \Sigma^n$, if $d(c, y) \leq \rho n$, then find an efficient algorithm to list all the $c \in B_{\rho n}(y) \cap C$, where $|B_{\rho n}(y) \cap C| = L$.



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- Some current cryptographic protocols based on IOPPs (e.g. protocol STIR in CRYPTO'24) used the list decodability of Reed–Solomon and related codes, which are fundamental in the theory of zero-knowledge proofs.

List Decodability

Definition (Combinatorial list decodability)

For $\rho \in [0,1]$ and $L \geq 1$, a code $C \subseteq \Sigma^n$ is (ρ, L) list decodable if for all $y \in \Sigma^n$ and $L+1$ distinct codewords $c_0, c_1, \ldots, c_L \in C$, we have $\max_{0 \leq i \leq L} d(y, c_i) > \rho n$.

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Remark (Algorithmic list decoding)

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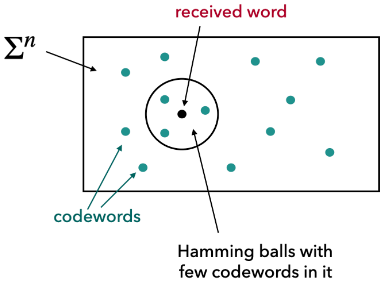

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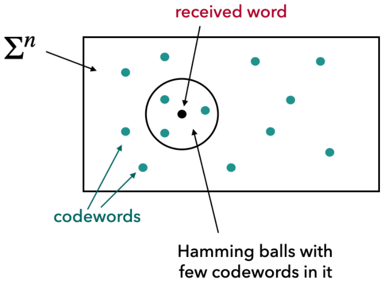
Combinatorics in List Decoding



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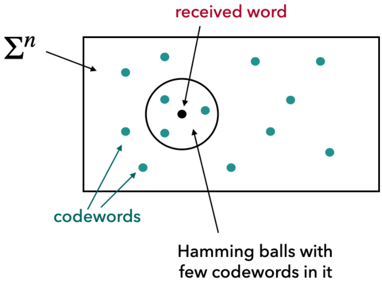

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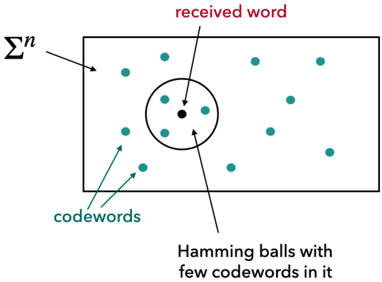

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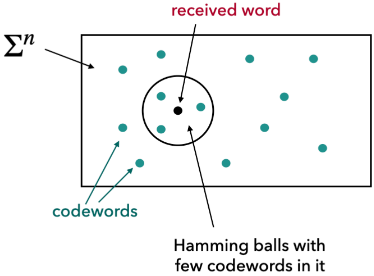
Sphere Packing: Given a code (subset or subspace of $\mathbb{F}_q^n$), we want to determine the best trade-off between the relative (Hamming) radius $\rho := d/n$, the rate $R = k/n$, and the list size L.



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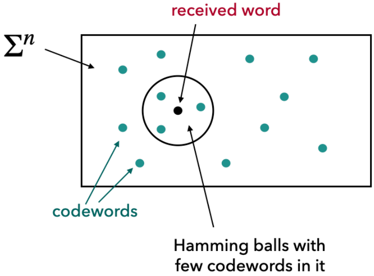

Core Challenge: Design such codes with efficient encoding and decoding algorithms!

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Proposition (Singleton bound)

For any $[n, k]$ linear code $C \subset \mathbb{F}_q^n$ we have $d(C) \leq n - k + 1$.

Reed–Solomon Code Codeword: An Example

Let $(m_1, m_2, m_3) \in \mathbb{F}_q^3$ be a message of length 3. Then the encoder of $\mathsf{RS}_{4,3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is below

$$(m_1, m_2, m_3) \xrightarrow{\text{encoder}} (f(\alpha_1), f(\alpha_2), f(\alpha_3), f(\alpha_4))$$

where $f(X) = m_1 + m_2 X + m_3 X^2$.

A First Bound for List Decodability

- Johnson Bound: Any code C with large distance $d(C) \xrightarrow{implies}$ (relatively) good list decodability: $\left(1 - \sqrt{1 - \frac{d(C)}{n}}, qnd(C)\right)$

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Theorem (List decoding capacity theorem)

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- *If a code of rate* R *and length* n *is* $(1 - R + \varepsilon, L)$ *list decodable code, then* $L \geq 2^{\Omega_\varepsilon(n)}$.

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Definition (Hasse derivative)

Given a finite field $\mathbb{F}_q, j \in \mathbb{N}$, and a polynomial $f(X)$, the j-th Hasse derivative $f^{(j)}(X)$ is defined as the coefficient of Z^j in the expansion

$$f(X+Z) = \sum_{i \in \mathbb{N}} f^{(j)}(X) Z^j.$$

More About Subinear Codes: Multiplicity Codes

Definition (Multiplicity codes)

An order-s multiplicity code $\mathrm{MULT}^{s}_{n,k,q}(\alpha_1,\ldots,\alpha_n)$ over $\mathbb{F}_q^s$ is defined as

$$\left\{ \begin{pmatrix} f(\alpha_1) & f(\alpha_2) & \cdots & f(\alpha_n) \\ f^{(1)}(\alpha_1) & f^{(1)}(\alpha_2) & \cdots & f^{(1)}(\alpha_n) \\ \vdots & \vdots & \cdots & \vdots \\ f^{(s-1)}(\alpha_1) & f^{(s-1)}(\alpha_2) & \cdots & f^{(s-1)}(\alpha_n) \end{pmatrix} : \substack{f(X)\in\mathbb{F}_q[X] \\ \deg(f)<k} \right\} \subseteq \left(\mathbb{F}_q^s\right)^n$$

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Remark: $d(\text{MULT}^{s}_{n,k,q}(\alpha_1,\ldots,\alpha_n)) = sn - k + 1.$

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- **Barrier:** For folded RS codes of rate R and block length n, the best known list-decoding radius is $1 - R - \varepsilon$, but the list size of Guruswami–Rudra (STOC'06) is $n^{O(1/\varepsilon)}$.

Open Problem (Guruswami–Rudra'06)

It remains an open question to reduce this list size $n^{O(1/\varepsilon)}$*, given that existential random coding arguments work with a list size of* $O(1/\varepsilon)$.

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The Open Problem of Guruswami–Rudra at STOC'06

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We fully resolved the near two-decade-old open problem of Guruswami and Rudra (STOC'06).

Our Results

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Theorem (Chen–Zhang STOC'25)

For $L \geq 1$, any appropriate folded RS codes $\mathrm{FRS}_{n,k,q,s}(\alpha_1, \ldots, \alpha_n)$ *of rate $R := k/n$ and block length n is* $\left(\frac{L}{L+1}\left(1 - \frac{sR}{s-L+1}\right), L\right)$ *average-radius list decodable.*

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Remark (Concurrent and Independent Work)

In a concurrent and independent work, Srivastava (SODA'25) shows the $\left(\frac{L}{L+1}\left(1 - \frac{sR}{s-L+1}\right), L^2\right)$ list-decodablility — a weaker result — for folded RS codes.

More about Our Results

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This also yields an exponential improvement over the previous state-of-the-art by Kopparty, Ron-Zewi, Saraf, and Wootters (FOCS'18), whose approach requires a list size of $(1/\varepsilon)^{O(1/\varepsilon)}$.

Generalized Singleton Bound

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Theorem (Generalized Singleton bound, Shangguan–Tamo STOC'20)

For any code C of rate R, if C is (ρ, L) list decodable and $q > L$, then

$$\rho \lesssim \frac{L}{L+1}(1-R).$$

FRS Proof Overview: The First Step

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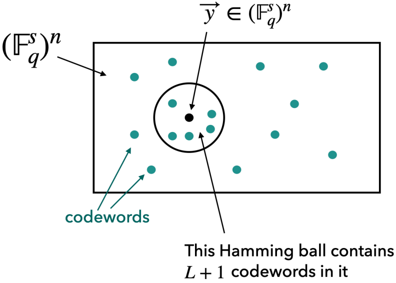

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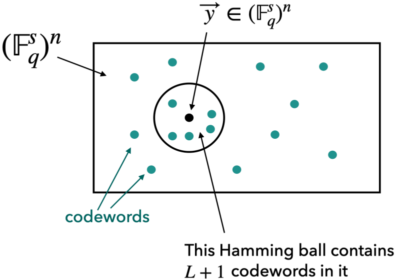

There exist a point $\vec{y} \in (\mathbb{F}_q^s)^n$ and $L+1$ pair-wise distinct codewords $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{L+1} \in \mathsf{FRS}_{n,k,q,s}(\alpha_1, \ldots, \alpha_n)$ such that

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$$\sum_{i=1}^{L+1} I(\vec{y}, \vec{c}_i) \geq n + \frac{Lk}{s-L+1} \qquad (*)$$

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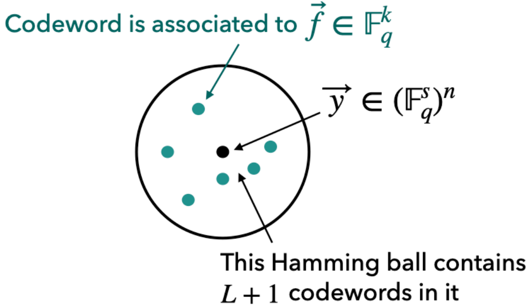

New Perspectives: Geometric Agreement Hypergraph

Consider the folded RS code $\mathsf{FRS}_{n,k,q,s}(\alpha_1, \ldots, \alpha_n)$, a received word $\vec{y} \in (\mathbb{F}_q^s)^n$, and ℓ vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_\ell \in \mathbb{F}_q^k$.

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Definition (Geometric agreement hypergraph based on FRS codes)

We define the geometric agreement hypergraph $(\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} := \left\{\vec{f_1}, \vec{f_2}, \ldots, \vec{f_\ell}\right\}$ and a tuple of n hyperedges $\mathcal{E} := \{e_1, e_2, \ldots, e_n\}$, where $e_i := \left\{\vec{f_j} \in \mathcal{V} : \vec{y}[i] = \mathrm{Enc}(f_j)[i]\right\}$.

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We define the weight $\mathsf{wt}(\mathcal{V}, \mathcal{E}) := \sum_{i=1}^{n} \mathsf{wt}(e_i)$, where $\mathsf{wt}(e_i) := \max(|e_i| - 1, 0)$.

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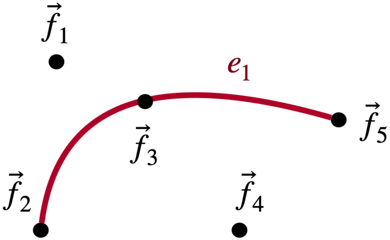
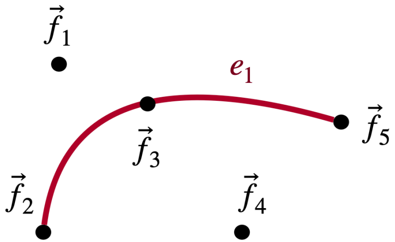
A lot of "agreements" implies the corresponding geometric agreement hypergraph has large enough weight.

Geometric Agreement Hypergraph: An Example





Consider an example when $\ell = 5$. Given $\vec{y} = (y_1, y_2, \ldots, y_n)$ in $(\mathbb{F}_q^s)^n$, the red hyperedge $e_1 = \{\vec{f}_2, \vec{f}_3, \vec{f}_5\}$ tells us that

$$\begin{aligned} y_1 &= \mathrm{Enc}(f_2)[1] := (f_2(\alpha_1), f_2(\gamma\alpha_1), \ldots, f_2(\gamma^{s-1}\alpha_1))^\top \\ &= \mathrm{Enc}(f_3)[1] := (f_3(\alpha_1), f_3(\gamma\alpha_1), \ldots, f_3(\gamma^{s-1}\alpha_1))^\top \\ &= \mathrm{Enc}(f_5)[1] := (f_5(\alpha_1), f_5(\gamma\alpha_1), \ldots, f_5(\gamma^{s-1}\alpha_1))^\top. \end{aligned}$$

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Let $f_1(X), \ldots, f_h(X) \in \mathbb{F}_q[X]$ and $\gamma \in \mathbb{F}_q^\times$. We define their γ-folded Wronskian $W_\gamma(f_1, \ldots, f_h)(X) \in (\mathbb{F}_q[X])^{h \times h}$ by

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Lemma (Folded Wronskian criterion for linear independence, Guruswami–Kopparty FOCS'13)

Let $k < q$ and $\vec{f}_1, \ldots, \vec{f}_h \in \mathbb{F}_q^k$. Let γ be a generator of $\mathbb{F}_q^\times$. Then $\vec{f}_1, \ldots, \vec{f}_h$ are linearly independent over $\mathbb{F}_q$ if and only if the folded Wronskian determinant $\det W_\gamma(f_1, \ldots, f_h)(X) \neq 0$.

Object: For a given geometric agreement hypergraph, we can define a corresponding geometric polynomial based on the determinant of folded Wronskian.

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Definition (Geometric polynomial)

Given L non-zero vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_L \in \mathbb{F}_q^k$ such that $\dim_{\mathbb{F}_q}(\mathrm{Span}_{\mathbb{F}_q}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_L\}) = \ell \in [L]$.

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$$\lambda_{i_1, i_2, \ldots, i_\ell} \cdot \det W_\gamma(f_{i_1}, \ldots, f_{i_\ell})(X),$$

where $\lambda_{i_1, i_2, \ldots, i_\ell} \in \mathbb{F}_q^\times$ and $\{f_{i_1}, \ldots, f_{i_\ell}\}$ forms a $\mathbb{F}_q$-basis of the space $\mathrm{Span}_{\mathbb{F}_q}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_L\}$.

Geometric Agreement Hypergraph Provides Zeros of a Geometric Polynomial With Multiplicity

For $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m \in \mathbb{F}_q^k$, we define (informally) $\widetilde{\dim}_{\mathbb{F}_q}\left(\vec{f}_1, \ldots, \vec{f}_m\right)$ as the dimension of the smallest affine subspace that contains all these vectors.

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Theorem (Alternatively stated in Guruswami–Kopparty FOCS'13)

Given L distinct non-zero $\vec{f}_1, \vec{f}_2, \dots, \vec{f}_L \in \mathbb{F}_q^k$. Let $(\mathcal{V}, \mathcal{E})$ be a geometric agreement hypergraph over $\mathcal{V} = \left\{0, \vec{f}_1, \dots, \vec{f}_L\right\}$ where $\mathcal{E} = \left\{e_1, \dots, e_n \subseteq \mathcal{V}\right\}$, then $V_{\{f_i\}_{i \in L}}(X)$ has at least

$$(s - \ell + 1) \sum_{i=1}^{n} \widetilde{\dim}_{\mathbb{F}_q}(e_i)$$

roots with multiplicity, where $\dim(\mathrm{Span}_{\mathbb{F}_q}\{\vec{f}_1, \dots, \vec{f}_L\}) = \ell$.

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$$(*) \quad \xrightarrow{\text{implies}} \quad \mathsf{wt}(\mathcal{V}, \mathcal{E}) := \sum_{i=1}^{n} \max(|e_i| - 1, 0) \geq \frac{Lk}{s - L + 1}$$

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We do NOT have much loss!

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Let $\{\vec{f}_i\}_{i\in[L]}$ *be a set of distinct non-zero vectors in* $\mathbb{F}_q^k$ *and vertices* $\mathcal{V} := \{0, \vec{f}_1, \ldots, \vec{f}_L\}$. *Let* $\dim_{\mathbb{F}_q}(\mathrm{Span}_{\mathbb{F}_q}\{\vec{f}_1, \ldots, \vec{f}_L\}) = \ell$.

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$$\sum_{i\in[n]} \mathrm{Loss}(e_i) \leq \frac{(L-\ell)k}{s-L+1}.$$

Open Problems and Future Directions

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The End

Questions?